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# Three-Particle Regge Pole in $\phi^3$ Theory*

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## ABSTRACT

We prove that the location $-2 + \alpha_3(\vec{\Delta})$ of the three-particle Regge pole in $\phi^3$ theory obeys

$$\alpha_3(0) > 2.2292714924\ \alpha(0)$$

where $-2 + \alpha(0)$ is the position of the Reggeon-particle cut.

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In the previous paper[1] on the high energy behavior of $\phi^3$ theory we demonstrated that in addition to the usual Mandelstam diagrams which lead to the "Reggeon-particle" cut located at (we take $m = 1$)

$$-2 + \alpha(0) = -2 + \frac{g^2}{16\pi^3}\pi \tag{1}$$

there is a much larger class of diagrams which are equally important. We also showed that when all these diagrams are included the full amplitude has a three-particle Regge pole at $-2 + \alpha_3(\vec{\Delta})$ where

$$\alpha_3(\vec{\Delta}) = \sup_h \Bigg\{ \Bigg[ \int d^2\vec{k}\, h^2(\vec{k})\,\alpha(\vec{k}) + 2\left(\frac{g^2}{16\pi^3}\right)\int d^2\vec{k}\, d^2\vec{k}'\, h(\vec{k})h(\vec{k}')(\vec{k}^2+1)^{-\frac{1}{2}}(\vec{k}'^2+1)^{-\frac{1}{2}}[(\vec{\Delta}-\vec{k}-\vec{k}')^2+1]^{-1}\Bigg] \left[d^2\vec{k}\, h^2(\vec{k})\right]^{-1}\Bigg\} \tag{2}$$

with

$$\alpha(\vec{k}) = \frac{g^2}{16\pi^3}\int d^2\vec{k}'\, \frac{1}{\vec{k}'^2+1}\, \frac{1}{(\vec{k}-\vec{k}')^2+1} . \tag{3}$$

The purpose of this note is to show that

$$\alpha_3(0) > 2.2292714924\, \alpha(0) . \tag{4}$$

To prove (4) we choose in (2)

$$h(\vec{k}) = (\vec{k}^2+1)^{-3/2}. \tag{5}$$

Then, for convenience, define J to be the right-hand side of (2) evaluated with (5) divided by $2\alpha(0)$. Explicitly, since

$$\int d^2\vec{k}\; h^2(\vec{k}) = \frac{\pi}{2} \tag{6}$$

we have

$$J = \pi^{-2}\left\{\int\!\!\int d^2\vec{k}\; d^2\vec{k}'\; \frac{1}{(\vec{k}^2+1)^3}\;\frac{1}{\vec{k}^2+1}\;\frac{1}{(\vec{k}-\vec{k}')^2+1}\right.$$

$$\left. + 2\int d^2\vec{k}\; d^2\vec{k}'\; \frac{1}{(\vec{k}^2+1)^2}\;\frac{1}{(\vec{k}'+1)^2}\;\frac{1}{(k-k')^2+1}\right\}. \tag{7}$$

Define the two-dimensional Fourier transform by

$$\bar{F}(\vec{\zeta}) = (2\pi)^{-1}\int d^2\vec{k}\; e^{i\vec{\zeta}\cdot\vec{k}}\; F(\vec{k}) \quad . \tag{8}$$

Then, using

$$\int d^2\vec{k}\; A(\vec{k})B(\vec{k}) = \int d^2\vec{\zeta}\; \bar{A}(\zeta)\bar{B}(\zeta) \tag{9}$$

and

$$\frac{1}{2\pi}\int d^2\vec{k}\; d^2\vec{k}'\; A(k')B(k-k')e^{i\vec{\zeta}\cdot\vec{k}} = 2\pi\bar{A}(\vec{\zeta})\bar{B}(\vec{\zeta}) \tag{10}$$

and defining

$$\bar{F}_j(\zeta) = \frac{1}{2\pi}\int d^2\vec{k}\; \frac{e^{i\vec{\zeta}\cdot\vec{k}}}{(\vec{k}^2+1)^j} \tag{11}$$

we have

$$J = 4\int_0^\infty d\zeta\,\zeta\,[\bar{F}_3(\zeta)\bar{F}_1(\zeta)^2 + 2\bar{F}_2(\zeta)\bar{F}_1(\zeta)] \quad . \tag{12}$$

Now

$$\frac{1}{2\pi}\int d^2\vec{k}\ \frac{e^{i\vec{k}\cdot\vec{\zeta}}}{\vec{k}^2 + a^2} = K_0(a\zeta) \tag{13}$$

when $K_0(\zeta)$ is the modified Bessel function of the second kind. Thus we find by repeated differentiation that

$$F_1(\zeta) = K_0(\zeta) \tag{14}$$

$$F_2(\zeta) = -\frac{\zeta}{2}K_0'(\zeta) \tag{15}$$

and

$$F_3(\zeta) = \frac{\zeta^3}{8}\frac{\partial}{\partial\zeta}\zeta^{-1}\frac{\partial}{\partial\zeta}K_0(\zeta) \quad . \tag{16}$$

Using the differential equation[2] for $K_0$

$$K_0'' + \zeta^{-1}K_0' - K_0 = 0 \tag{17}$$

we rewrite (16) as

$$F_3(\zeta) = \frac{1}{8}\zeta^2\left[-2\zeta^{-1}K_0' + K_0\right] \quad . \tag{18}$$

Substituting (14), (15) and (18) into (12) yields

$$J = \frac{1}{2}\int_0^\infty d\zeta\ \zeta^3\left\{(-2\zeta^{-1}K_0' + K_0)K_0^2 + 2K_0'(K_0^2)'\right\} \quad . \tag{19}$$

Integrate the last term by parts to obtain

$$J = -\frac{1}{2}\int_0^\infty d\zeta\ \zeta^2K_0^2\,(6K_0' + K_0) \quad . \tag{20}$$

Then, if we integrate the first term by parts and define

$$I_1 = \int_0^\infty d\zeta\ \zeta K_0(\zeta)^3 \tag{21}$$

and

$$I_2 = \int_0^\infty d\zeta\ \zeta^3 K_0(\zeta)^3 \tag{22}$$

we find

$$J = 2I_1 - \frac{1}{2} I_2 \quad . \tag{23}$$

To proceed further we use Nicholson's formula[3]

$$K_0(\zeta)^2 = 2 \int_0^\infty dt\ K_0(2\zeta \cosh t) \tag{24}$$

and the integral[4]

$$2^{\rho+2}\ \Gamma(1-\rho) \int_0^\infty dt\ K_0(\alpha t) K_0(t) t^{-\rho}$$

$$= \alpha^{\rho-1}\, F(\tfrac{1}{2} - \tfrac{1}{2}\rho, \tfrac{1}{2} - \tfrac{1}{2}\rho; 1-\rho; 1-\alpha^{-2})\, \Gamma(\tfrac{1}{2} - \tfrac{1}{2}\rho)^4 , \tag{25}$$

where F(a, b;c;z) is the hypergeometric function. We find

$$I_1 = \int_0^\infty dt\,(2\cosh t)^{-2}\, F\left[1, 1;2;1-(2\cosh t)^{-2}\right] \tag{26}$$

and

$$I_2 = \frac{2}{3}\int_0^\infty dt\,(2\cosh t)^{-4}\, F\left[2, 2;1 - (2\cosh t)^{-2}\right]. \tag{27}$$

Specifically,[5] we have

$$F(1, 1;2;z) = -z^{-1}\ln(1-z)$$

and

$$F(2, 2;4;z) = 6\left\{\left[-2z^{-3} + z^{-2}\right]\ln(1-z) - 2z^{-2}\right\}. \tag{28}$$

Therefore

$$I_1 = 2\int_0^\infty dt\,\frac{\ln(2\cosh t)}{(2\cosh t)^2 - 1} \tag{29}$$

and

$$I_2 = 8\int_0^\infty dt\,\left[(2\cosh t)^2 - 1\right]^2\left[\frac{(2\cosh t)^2 + 1}{(2\cosh t)^2 - 1}\ln(2\cosh t) - 1\right]. \tag{30}$$

Make the change of variable

$$x = e^{2t} \tag{31}$$

to obtain

$$I_1 = \frac{1}{2}\int_0^\infty dx \, \frac{\ln(x+1)}{(x+1)^2 - x} \tag{32}$$

and

$$I_2 = 2\int_0^\infty dx \, \frac{x}{[(1+x)^2 - x]^2}\left\{\frac{(x+1)^2 + x}{(x+1)^2 - x}\ln(1+x) - 1\right\}. \tag{33}$$

Then let

$$x = \frac{1}{y} - 1 \tag{34}$$

and find

$$I_1 = -\frac{1}{2}\int_0^1 dy \, \frac{\ln y}{y^2 - y + 1} \tag{35}$$

and

$$I_2 = -2\int_0^1 dy \, y(1-y)\frac{1}{(1-y+y^2)^2}\left[\frac{1+y-y^2}{1-y+y^2}\ln y + 1\right]. \tag{36}$$

Next define

$$z = 2y - 1. \tag{37}$$

Then

$$I_1 = -\int_0^1 dz \, \frac{\ln \frac{1}{4}(1-z^2)}{3+z^2} \tag{38}$$

and

$$I_2 = -\frac{4}{3}\int_0^1 dz\left\{\frac{\ln \frac{1}{4}(1-z^2)}{3+z^2} + 2\,\frac{3-z^2}{(3+z^2)^2}\right\}. \tag{39}$$

Therefore we find

$$I_2 = \frac{4}{3} I_1 - \frac{2}{3} \tag{40}$$

and hence

$$J = \frac{4}{3} I_1 + \frac{1}{3} = I_2 + 1 \, . \tag{41}$$

By expanding (35) as

$$I_1 = -\frac{1}{2}\int_0^1 dy \, \frac{1 + y}{1 + y^3} \ln y = -\frac{1}{2}\int_0^1 dy \, (1 + y) \ln y \sum_{n=0}^{\infty} (-1)^n y^{3n}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \left[ (3n+1)^{-2} + (3n+2)^{-2} \right] \tag{42}$$

and using[6]

$$\sum_{n=0}^{\infty} (z + n)^{-2} = \psi'(z) \, , \tag{43}$$

where $\psi(z)$ is the logarithmic derivative of the gamma function, we have

$$I_1 = \frac{1}{72} \left\{ \psi'\left(\frac{1}{6}\right) + \psi'\left(\frac{1}{3}\right) - \psi'\left(\frac{1}{2}\right) - \psi'\left(\frac{5}{6}\right) \right\} . \tag{44}$$

Numerically we find

$$I_1 = 0.5859768097 \tag{45a}$$

and

$$I_2 = 0.1146357462 \, . \tag{45b}$$

Using (45) in (41) and using the definition of J (4) follows.

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[2] Bateman Manuscript Project, Higher Transcendental Functions, edited by A. Erdelyi (McGraw-Hill Book Co., Inc., New York, 1953) Vol. 2, p. 5.

[3] Ibid., Vol. 2, p. 54, Eq. (39).

[4] Ibid., Vol. 2, p. 93, Eq. (36).

[5] Ibid., Vol. 1, p. 102, Eqs. (15), (20), and (24).

[6] Ibid., Vol. 1, p. 22, Eq. (22).